Complex numbers and planar geometry

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① complex numbers $\mathbb{C}$: basics

want: $\sqrt{-1} =: i$

formally: $\boxed{z = x + i \cdot y}$, $x \in \mathbb{R}$
$y \in \mathbb{R}$

addition: $(x_1 + i \cdot y_1) + (x_2 + i \cdot y_2)$

$$\overset{\text{def.}}{=} (x_1 + x_2) + i \cdot (y_1 + y_2)$$

multiplication: $(x_1 + i \cdot y_1) \cdot (x_2 + i \cdot y_2)$

$$= x_1 x_2 + i \cdot (y_1 x_2 + y_2 x_1) + \underbrace{i^2}_{(-1)} \cdot y_1 y_2$$

$$\overset{\text{def.}}{=} (x_1 x_2 - y_1 y_2) + i \cdot (y_1 x_2 + y_2 x_1)$$

$\mathbb{C} = \{ x + i \cdot y \mid x, y \in \mathbb{R} \}$ has good prop.:

(1) $(\mathbb{C}, +, \cdot)$ is a field.

fundamental theorem of algebra

of degree ≥ 1 has a root $\xi \in \mathbb{C}$.

(i.e. $P(\xi) = 0$).

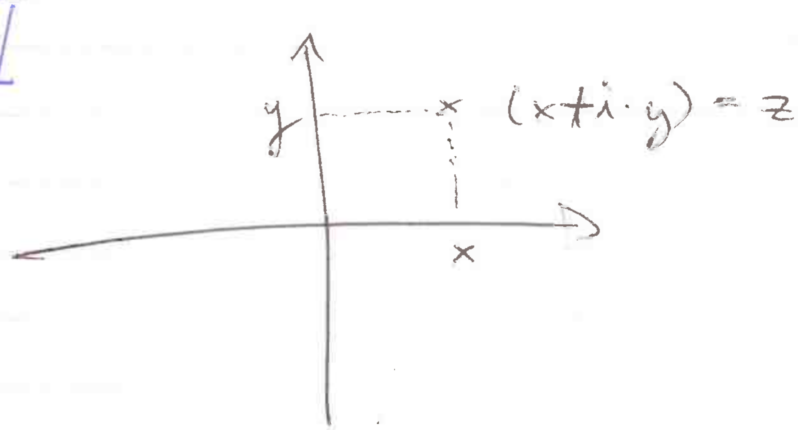
example: $\boxed{P(z) = (z^2 + 1)}$.

$\xi = i$.

More interesting for us:

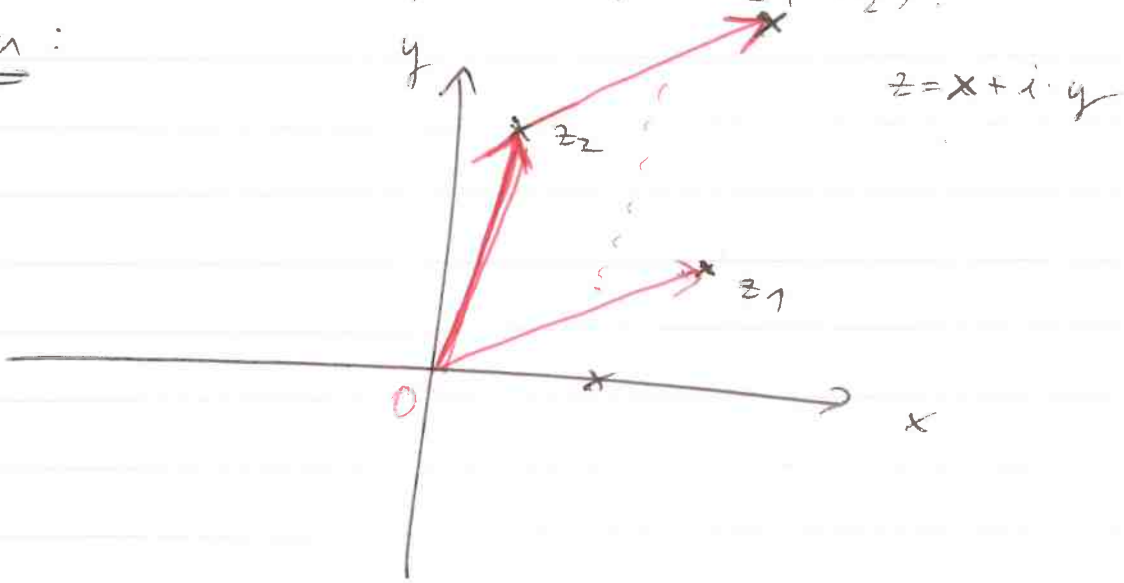
$(+, \cdot)$ $\mathbb{C} = \mathbb{R} + i \cdot \mathbb{R} \approxeq \mathbb{R}^2$ (x, y) $+$ multiplication?

$\boxed{z = x + iy}$

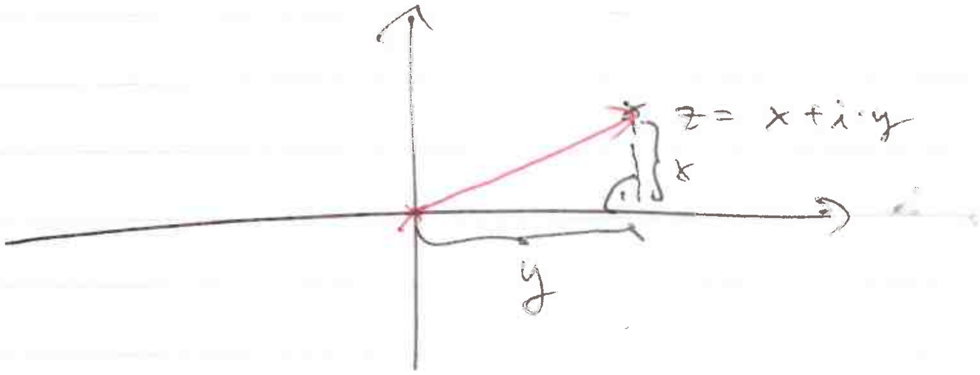


② complex numbers: geometry

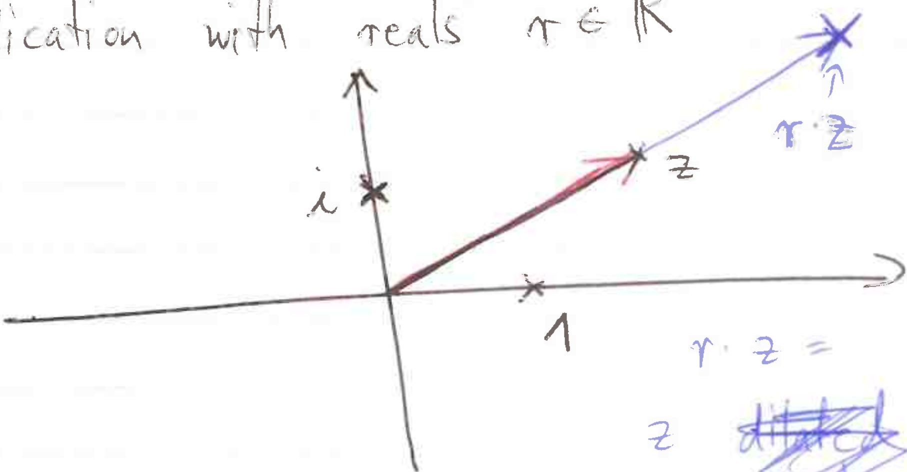
<u>addition</u>:



<u>absolute value</u>: $|x+i\cdot y| = \sqrt{x^2+y^2}$



multiplication with reals $r \in \mathbb{R}$



$r \cdot z =$

z ~~dilated~~ ~~streck~~ ~~street~~ stretched by factor r

~~That~~ "polar coordinates"

$$0 \neq z = \underset{\substack{\Uparrow \\ \mathbb{R}}}{|z|} \cdot \underbrace{\boxed{\left(\frac{z}{|z|}\right)}}_{=:u} . \qquad |u| = 1$$

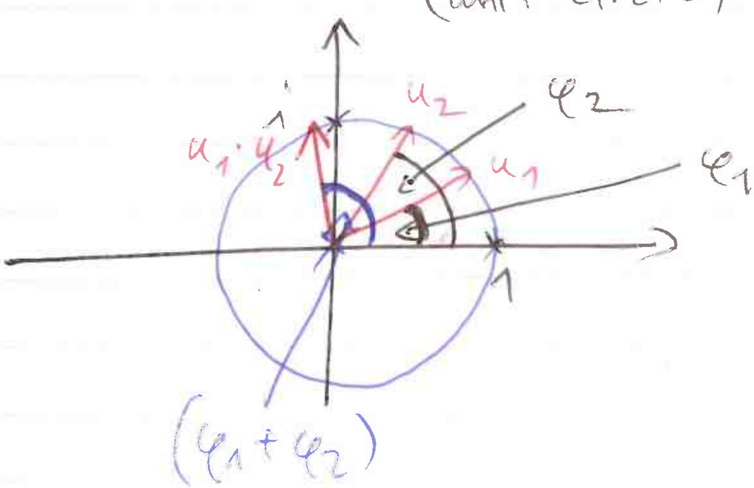
Since multiplication is commutative,

$$z_1 \cdot z_2 = \left(|z_1| \cdot |z_2|\right) \cdot \left(\underbrace{\frac{z_1}{|z_1|}}_{=:u_1} \cdot \underbrace{\frac{z_2}{|z_2|}}_{=:u_2} \right).$$

we only have to understand

$$u_1 \cdot u_2 , \quad u_1, u_2 \in \mathbb{C}, \ |u_1| = |u_2| = 1 .$$

Set $S^1 := \{ u \in \mathbb{C} \mid |u| = 1 \}$.
(unit circle)



Proof: ~~un~~ Every $u \in S^1$ can be written

$$u = \cos(\varphi) + i \cdot \sin(\varphi) .$$

$$u_1 \cdot u_2 = (\cos(\varphi_1) + i \cdot \sin(\varphi_1)) \cdot (\cos(\varphi_2) + i \cdot \sin(\varphi_2))$$

$$= (\cos(\varphi_1) \cdot \cos(\varphi_2) - \sin(\varphi_1) \cdot \sin(\varphi_2)) + i \cdot (\sin(\varphi_1) \cdot \cos(\varphi_2) + \cos(\varphi_1) \cdot \sin(\varphi_2))$$

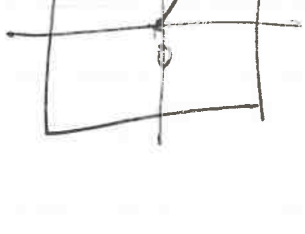
$$= (\cos(\varphi_1 + \varphi_2) + i \cdot \sin(\varphi_1 + \varphi_2))$$

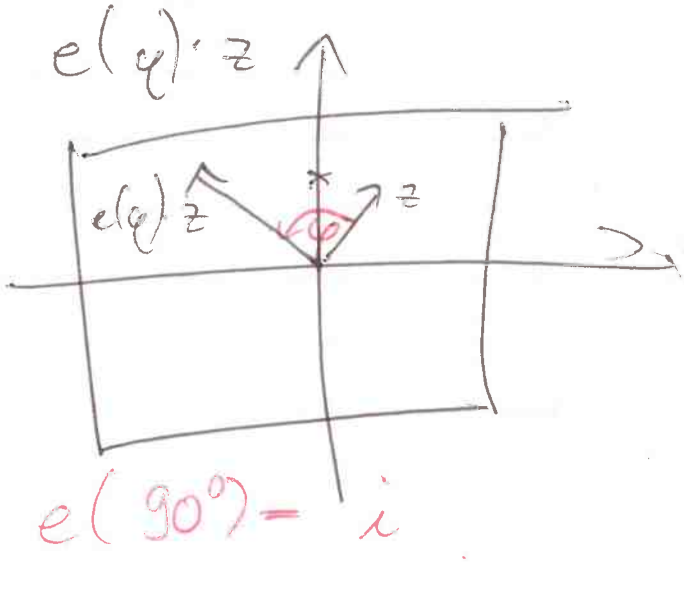
Def.: For angle φ define

$$e(\varphi) := \cos(\varphi) + i \cdot \sin(\varphi) \in \mathbb{C}$$

Multiplication by $e(\varphi)$ is rotation by φ:

$$z \in \mathbb{C} \rightsquigarrow e(\varphi) \cdot z$$

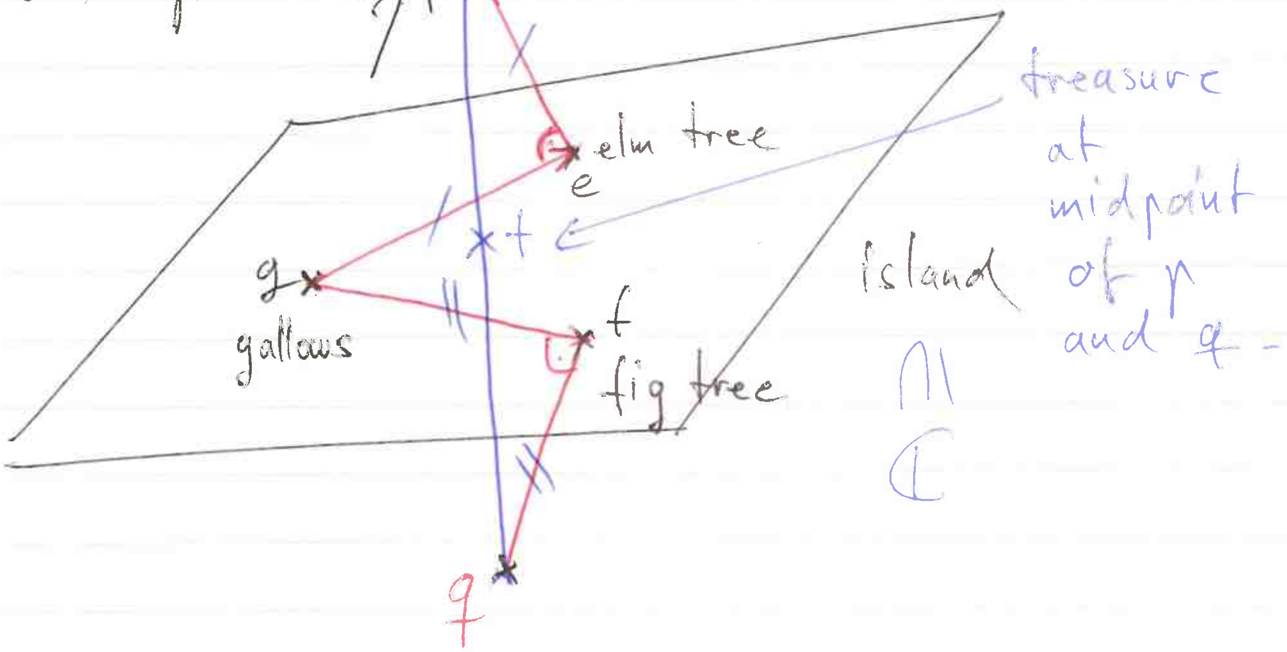




$\varphi = 90^\circ \implies e(90^\circ) = i$

3. Simple problems

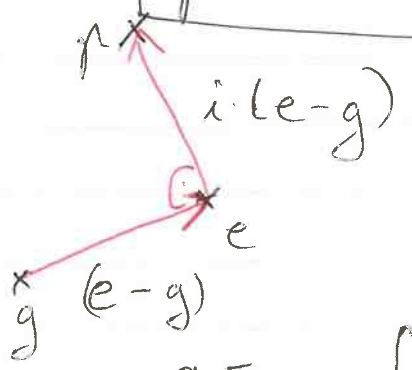
(1) pirate's treasure

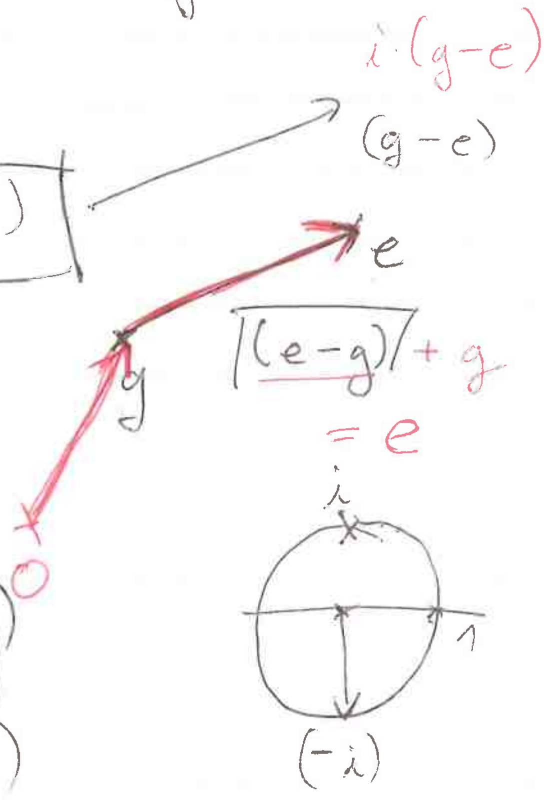


Suppose you know only where elm and fig tree are, but not the gallows.

Can you find t?

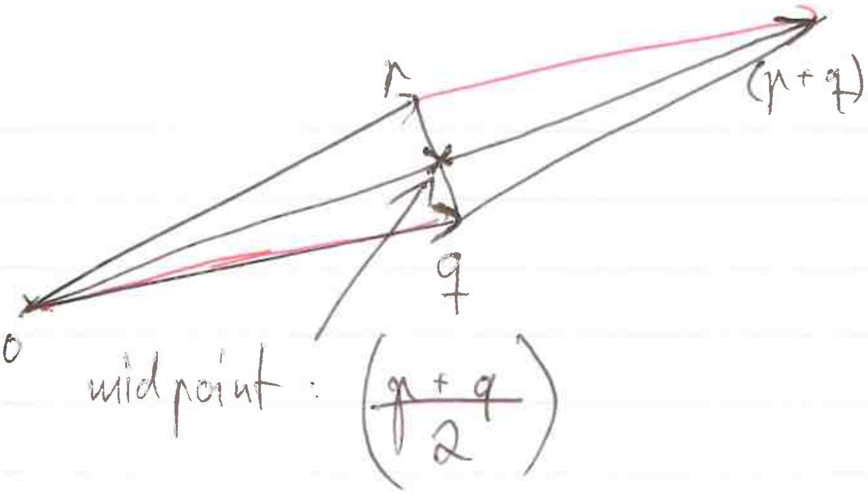
$$\boxed{p = e + i\cdot(e-g)}$$





$$q = f + e(270°)(f-g)$$

$$= f - i\cdot(f-g)$$



$$t = \frac{p+q}{2} = \frac{e+f+i(e-\cancel{g})-i(f-\cancel{g})}{2}$$

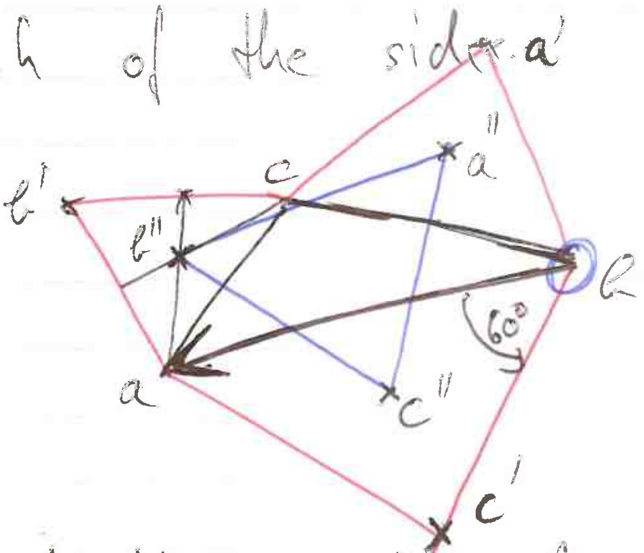
$$= \left(\frac{e+f}{2}\right) + i\cdot\left(\frac{e-f}{2}\right)$$

(2) napoleonic triangle

<u>Problem</u> : given $\triangle ABC$,

erect equilateral triangles outwards

on each of the sides



Prove that their centers form an

equilateral triangle.

Aside: Hamiton quaternions $\Rightarrow$ ...

not commutative

$$h = (x + i \cdot y + j \cdot v + k \cdot w)$$

Ebbinghaus et. al,
the Numbers

$x, y, v, w \in \mathbb{R}$

Set $\varepsilon := e(60^\circ)$.

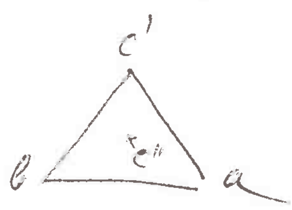
$$c' = b + \varepsilon \cdot (a - b)$$

$$a' = c' + \varepsilon \cdot (b - c)$$

$$b' = a + \varepsilon \cdot (c - a)$$

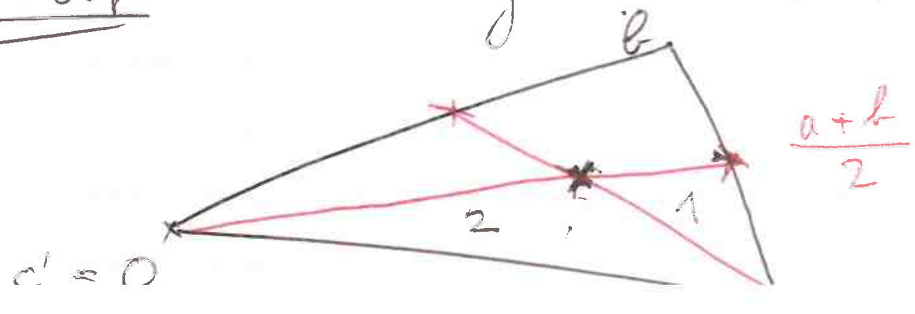
cyclic permutation: $a \to b \to c \to a$

centers: a'', b'', c''.



Claim: $c'' = \dfrac{a + b + c'}{3}$.

Proof: w.l.o.g.: $c' = 0$.

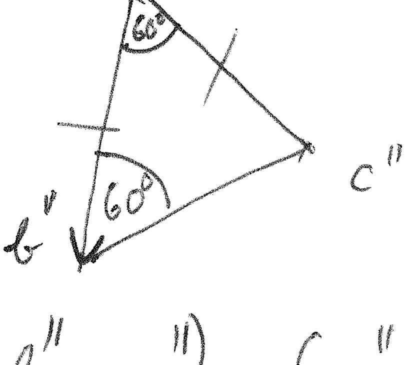


Fact: The medians in a triangle intersect in a $2:1$-ratio.

$$\text{center} = \left(\frac{a+b}{2}\right)\cdot\frac{2}{3} = \frac{a+b}{3}.$$

$\square$ claim.

$$3\cdot c'' = a + b + c'$$
$$= a + b + b + \varepsilon\cdot(a-b)$$
$$= (a+2b) + \varepsilon\cdot(a-b)$$
$$3\cdot a'' = (b+2c) + \varepsilon\cdot(b-c)$$
$$3\cdot b'' = (c+2a) + \varepsilon\cdot(c-a)$$



Want to prove: $\varepsilon\cdot(b''-a'') = (c''-a'')$.

$$3\varepsilon\cdot(b''-a'') = (\varepsilon)\cdot\Big((c+2a-b-2c) + \varepsilon\cdot(c-a-b+c)\Big)$$

$$= \varepsilon \cdot (2a - b - c)$$

$$+ \varepsilon^2 (2c - a - b)$$

$\ulcorner$ $\varepsilon^2 = ?$ $\quad \varepsilon = e(60^\circ)$

$$\varepsilon^3 = e(180^\circ) = (-1)$$

$$(\varepsilon^3 + 1) = 0$$

$$(\varepsilon + 1) \cdot \boxed{(\varepsilon^2 - \varepsilon + 1)} = 0 .$$

$$\varepsilon^2 - \varepsilon + 1 = 0$$

$$\varepsilon^2 = (\varepsilon - 1) . \quad \lrcorner$$

$$= \varepsilon \cdot (2a - b - c)$$

$$+ (\varepsilon - 1) \cdot (2c - a - b)$$

$$= (a + b - 2c) + \varepsilon \cdot (a - 2b + c) .$$

$$\left(= \varepsilon \cdot (b'' - a'') \right)$$

check: $= (c'' - a'')$.

$\square$ proof .